Topological Mixing in Hyperbolic Metric Spaces

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February 1, 2008

Abstract

If X is a δ-hyperbolic metric space in the sense of Gromov and Γ a non-elementary discrete group of isometries acting properly discontinuously on X, it is shown that the geodesic flow on the quotient space $Y = X/\Gamma$ is topologically mixing, provided that the non-wandering set of the flow equals the whole quotient space of geodesics $GY := GX/\Gamma$ and geodesics in X satisfy certain uniqueness and convergence properties. In addition, ∂X is assumed to be connected and a counter example is given concerning the necessity of this assumption.

AMS 1991 Subject Classifications. **Primary:** 57M20;
Secondary: 53C23

*This research was supported in part by The University of the Aegean Research Unit Grant 470

1 Introduction

The extensive study of the geodesic flow, primarily on Riemannian manifolds, has been concerned, among other properties, with the establishment of topological transitivity and topological mixing. For compact manifolds of negative curvature, topological transitivity of the geodesic flow was proved by Anosov in [1]. Topological mixing, a stronger property, has been shown for the class of compact manifolds with non-positive curvature by P.Eberlein in [12]. In this paper we establish topological mixing of the geodesic flow in the class of spaces which are quotients of proper geodesic metric spaces, which are hyperbolic in the sense of Gromov, by non-elementary discrete group of isometries. This notion of hyperbolicity was introduced in [13]. Throughout this paper we will use the term *hyperbolic space* for a proper geodesic metric space which is δ-hyperbolic in the sense of Gromov for some $\delta \geq 0$. If Γ is a *non-elementary discrete* group of isometries (see section 1.2 below for definitions) of a hyperbolic space X we consider the quotient metric space $Y = X/\,\Gamma$. The action of Γ on X induces an isometric action of Γ on the space of geodesics GX which we show that is properly discontinuous. Hence, the space of geodesics GY is defined as the quotient metric space $GX/\,\Gamma$ (see section 1.2 below for details). We will use the letter p to denote both projections $X \to Y$ and $GX \to GY$. Recall that GX consists of all isometries $g : \mathbb{R} \to X$ and its topology is the topology of uniform convergence on compact sets. As the action of Γ on X is not necessarily free, observe that an element $g \in GY$ is not a geodesic in the usual sense; it is just a continuous map $g : \mathbb{R} \to Y$ for which there exists an isometry $\overline{g} : \mathbb{R} \to X$ such that $g = p \circ \overline{g}$. The geodesic flow on X is defined by the

map

$$\mathbb{R} \times GX \rightarrow GX$$

where the action of $\mathbb{R}$ is given by right translation, i.e. for all $t \in \mathbb{R}$ and $g \in GX$, $(t, g) \rightarrow t \cdot g$ where $t \cdot g : \mathbb{R} \rightarrow X$ is the geodesic defined by $(t \cdot g)(s) = g(s + t), s \in \mathbb{R}$. If $t \in \mathbb{R}$ and $g \in GY$ define the geodesic flow on GY by setting

$$t \cdot g = p\big(t \cdot \overline{g}\big)$$

where $\overline{g}$ is any lift of g in GX.

Definition 1 *The geodesic flow* $\mathbb{R} \times GY \rightarrow GY$ *is topologically mixing if given any open sets* $\mathcal{O}$ *and* $\mathcal{U}$ *in* GY *there exists a real number* $t_0 > 0$ *such that for all* $|t| \geq t_0$, $t \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$.

A point g in GY belongs to the *non-wandering set* Ω of the geodesic flow $\mathbb{R} \times GY \rightarrow GY$ if there exist sequences $\{g_n\} \subset GY$ and $\{t_n\} \subset \mathbb{R}$, such that $t_n \rightarrow \infty$, $g_n \rightarrow g$ and $t_n \cdot g_n \rightarrow g$. Recall that a metric space is *geodesically complete* if each geodesic segment is the restriction of a geodesic defined on the whole real line. The main result of this paper is the following

Theorem 2 *Let* X *be a geodesically complete hyperbolic space and* Γ *a discrete group of isometries acting properly discontinuously on* X. *Assume that* ∂X *is connected,* X *has unique and convergent geodesics and* Γ *is non-elementary. Then the geodesic flow on the quotient space* $Y = X/\Gamma$ *is topologically mixing, provided that* $\Omega = GY$.

Uniqueness and convergence of geodesics are two properties imposed on X (called conditions (U) and (C)) which are standard for studying flows (cf [14]) and are explained in section 1.1 below. The assumption of geodesic completeness along with the connectivity condition on the boundary ∂X are imposed in order to assert existence of zeros for the generalized Busemann function α (defined in section 2 below), ie

$$\forall\ x, x' \in X, \exists\ \xi \in \partial X \ \text{ such that } \ \alpha\,(\xi, x, x') = 0 \tag{1}$$

This is what is really necessary for the proof of mixing presented in this paper and geodesic completeness along with connectivity of ∂X suffice for the above property to hold (see lemma 9 below). Note here that property (1) above follows also from the following property

$$\forall\ x \in X, \exists\ M > 0 : \left\{ y \in X \;\middle|\; d\,(x, y) = r \right\} \ \text{ is connected } \forall\ r > M \tag{2}$$

Hence, in the statement of theorem 2 above geodesic completeness as well as connectivity of ∂X can be replaced by property (2) alone.
The structure of this paper is as follows. In the present section we include basic definitions and properties of hyperbolic metric spaces. Moreover, the notion of non-elementary group Γ of isometries of X is explained and the action of such Γ on GX is analyzed. Finally, a counter-example is described in which neither properties (1), (2) hold, nor connectedness of ∂X. In section 2 Busemann functions are discussed and we use them to study strong stable sets in the space of geodesics. Although topological transitivity follows from topological mixing, we show in section 3 that the geodesic flow on Y is topologically transitive because we need this property in the proof of theorem 2 given in section 4. Finally, section 5 provides a class of spaces,

much wider than manifolds, satisfying all assumptions posited in theorem 2 above.

1.1 Preliminaries on hyperbolic metric spaces

The letter X will always denote a hyperbolic space i.e. a proper geodesic metric space which is δ-hyperbolic for some $\delta \geq 0$. For definitions and basic properties of hyperbolic spaces we refer the reader to [13], [11] and [3]. We recall here basic properties of the spaces GX and ∂X. GX consists of all isometric maps $g : \mathbb{R} \to X$ and its topology is the topology of uniform convergence on compact sets. In addition, we note here that GX is metrizable and the metric is given by the formula (see [13, Sect. 8.3])

$$d_{GX}(g_1, g_2) := \int_{-\infty}^{+\infty} e^{-|t|} d\big(g_1(t), g_2(t)\big)\, dt \tag{3}$$

If $g \in GX$ we will denote by $-g$ the geodesic defined by $(-g)(s) = g(-s)$ and, similarly, if $\mathcal{A} \subset GX$, then $-\mathcal{A} := \left\{ -g \;\middle|\; g \in \mathcal{A} \right\}$.

The boundary ∂X of a hyperbolic space can be defined as the space of equivalence classes of asymptotic geodesic rays starting at a fixed point in X. As X is assumed to be proper and geodesic this definition coincides with the definition of ∂X given using equivalence classes of sequences (cf [11, p.21]). If g is a geodesic, we will denote by $g(+\infty)$ the boundary point determined by the geodesic ray $g|_{[0,+\infty)}$ and similarly for $g(-\infty)$. We will use several results from [11, Ch.11 §1] which we have gathered in the following proposition.

Proposition 3 *If (X, d) is a proper geodesic $\delta-$hyperbolic metric space, there exists a metric d_μ on X and a constant $C > 0$ such that*

(a) $d_\mu(x,y) \le C\, d(x,y)$ *for all* $x, y \in X$.

(b) if $\partial_\mu X$ *is the boundary of the completion of* X *with respect to* d_μ*, then the completion* $\partial_\mu X \cup X$ *is a compact metric space*

(c) there exists a homeomorphism $\partial_\mu X \cup X \to \partial X \cup X$ *which is the identity on* X.

We will impose three conditions on the hyperbolic space X. The first is that the boundary ∂X is connected. The role of this assumption was explained above (see also example 5). The other two conditions, called (U) and (C), which we impose on the hyperbolic space X, are standard for studying flows. Recall that two geodesic rays g_1, g_2 (or geodesics) are called *asymptotic* if $d\big(g_1(t), g_2(t)\big)$ is bounded for all $t \in \mathbb{R}^+$.

Condition (U) : We say that X has *unique geodesics* if for any two points $x_1, x_2 \in X \cup \partial X$ there exists a unique geodesic joining these points.

Condition (C) We say that X has *convergent geodesics* if for any two asymptotic geodesic rays (or geodesics) g_1, g_2 there exists a real number d such that

$$\lim_{t\to\infty} d\big(g_1(t), g_2(t+d)\big) = 0.$$

These two conditions hold, for example, in any simply connected, complete, locally compact geodesic metric space with curvature less than or equal to χ, $\chi < 0$. For such spaces, it is shown in proposition 20 below that condition (C) holds and see, for example, [8] for a proof that condition (U) holds.

As usual, set $\partial^2 X = \{(\xi, \eta) \in \partial X \times \partial X : \xi \neq \eta\}$. Condition (U) asserts

that the fiber bundle

$$\rho : GX \to \partial^2 X$$

given by $\rho(g) = \big(g(-\infty), g(+\infty)\big)$ has a single copy of $\mathbb{R}$ as fiber. It is shown in [5, Prop.4.8] that there exists a trivialization

$$H : GX \xrightarrow{\approx} \partial^2 X \times \mathbb{R} \tag{4}$$

of ρ such that the conjugation of the geodesic flow with H is simply the map

$$(\xi_1, \xi_2, s) \to (\xi_1, \xi_2, s+t), \; for\ all \;\; (\xi_1, \xi_2) \in \partial^2 \widetilde{X} \;\; and \;\; s \in \mathbb{R}. \tag{5}$$

1.2 Quotient space by non-elementary group of isometries

We first recall the notion of non-elementary group of isometries. If X is a hyperbolic space and Γ a discrete group of isometries of X acting on X, the *limit set* $\Lambda(\Gamma)$ of the action of Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial X$, where x is arbitrary in X. The limit set has been studied extensively (see [10, ch. II], [11, ch. 2.1] for a detailed exposition) using the classification of the isometries of X into three types, namely, elliptic, parabolic and hyperbolic. If ϕ is hyperbolic then $\phi^n(x)$ converges to a point $\phi(+\infty) \in \partial X$ (resp. $\phi(-\infty) \in \partial X$) as $n \to +\infty$ (resp. $n \to -\infty$) with $\phi(+\infty) \neq \phi(-\infty)$. Moreover,

$$\begin{gathered} \forall\, \xi \in \partial X \setminus \{\phi(+\infty)\} \big(resp.\ \partial X \setminus \{\phi(-\infty)\}\big) \Longrightarrow \\ \phi^n(\xi) \to \phi(+\infty) \;\; \big(resp.\ \phi(-\infty)\big)\ as\ n \to \infty \;\; (resp.\ -\infty) \end{gathered} \tag{6}$$

The cardinality of the limit set is 0,1,2 or infinite. A group Γ acting on a hyperbolic space X is said to be *non-elementary* if the cardinality of $\Lambda(\Gamma)$ is infinite. In this case, the following result is shown in [10]:

$$\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma \ \textit{is hyperbolic}\} \ \textit{is dense in}\ \Lambda(\Gamma) \times \Lambda(\Gamma) \tag{7}$$

Note here that, as X is assumed to be proper, discreteness of the group Γ is equivalent to requiring that Γ acts properly discontinuously on X, ie for any compact $K \subset X$ the set $\left\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\right\}$ is finite (see [16, Th. 5.3.5]). We next analyze the action of Γ on GX in more detail.

Proposition 4 *Let Γ be a group of isometries of X acting properly discontinuously on X. Then Γ acts by isometries and properly discontinuously on the space of geodesics GX.*

Proof We have assumed that Γ acts by isometries on X. Therefore, if $f, g \in GX$ and $\gamma \in \Gamma$ we have

$$\int_{-\infty}^{+\infty} e^{-|t|} d(\gamma f(t), \gamma g(t))\, dt = \int_{-\infty}^{+\infty} e^{-|t|} d(f(t), g(t))\, dt$$

which implies that $d_{GX}(\gamma f, \gamma g) = d_{GX}(f, g)$. This shows that Γ acts on GX by isometries. Moreover, we have assumed that Γ acts properly discontinuously on X, i.e.

$$\forall \ \textit{compact}\ K \subset X, \left\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\right\} \ \textit{is finite} \tag{8}$$

We proceed to show that Γ acts properly discontinuously on GX. Let $\mathcal{K}$ be an arbitrary compact set in GX. Set $d = diam(\mathcal{K})$ and choose $g \in \mathcal{K}$

arbitrary. Using the triangle inequality in X one can show that

$$d\big(\gamma g\left(0\right), g\left(0\right)\big) - 2\left|t\right| \leq d\big(\gamma g\left(t\right), g\left(t\right)\big) \leq d\big(\gamma g\left(0\right), g\left(0\right)\big) + 2\left|t\right|$$

from which it follows, after integration, that

$$d\big(\gamma g\left(0\right), g\left(0\right)\big) - 4 \leq d_{GX}\left(\gamma g, g\right) \leq d\big(\gamma g\left(0\right), g\left(0\right)\big) + 4 \tag{9}$$

As the space X is assumed to be proper, the closure of the ball $B = B\big(g\left(0\right), 2d+4\big)$ centered at $g\left(0\right)$ and radius $2d+4$ is compact and, by (8), the set

$$A = \left\{\gamma \in \Gamma \;\middle|\; \gamma\overline{B} \cap \overline{B} \neq \emptyset\right\} \; \textit{is finite}$$

This together with equation (9) implies that for all but a finite number of elements $\gamma \in \Gamma$,

$$d_{GX}\left(\gamma g, g\right) \geq d\big(\gamma g\left(0\right), g\left(0\right)\big) - 4 > 2d + 4 - 4 = 2d$$

Let now f be arbitrary element of $\mathcal{K}$. Then, since $d_{GX}\left(f, g\right) < d = diam\left(\mathcal{K}\right)$ we have that for all but a finite number of elements $\gamma \in \Gamma$,

$$\begin{aligned} d_{GX}\left(\gamma f, g\right) &\geq \left|d_{GX}\left(\gamma f, \gamma g\right) - d_{GX}\left(\gamma g, g\right)\right| \\ &= d_{GX}\left(\gamma g, g\right) - d_{GX}\left(\gamma f, \gamma g\right) \\ &> 2d - d = d \end{aligned}$$

which implies that

$$\forall\, f \in \mathcal{K} \Rightarrow \gamma f \notin \mathcal{K}$$

for all but a finite number of elements $\gamma \in \Gamma$. In other words the set

$$\left\{\gamma \in \Gamma \;\middle|\; \gamma\mathcal{K} \cap \mathcal{K} \neq \emptyset\right\}$$

is finite. ∎

Define now GY to be the orbit space $\left\{ \Gamma g \mid g \in GX \right\}$ of the action of Γ on GX. The space GY can be viewed as the set of all continuous functions $g : \mathbb{R} \to Y$ for which there exists an isometry $\overline{g} : \mathbb{R} \to X$ satisfying $p \circ \overline{g} = g$. By abuse of language, we will be calling the elements of GY geodesics in Y.

Remark If X is a simply connected, complete, locally compact geodesic metric space with curvature less than or equal to χ, $\chi < 0$ and the action of Γ on X is free then GX/Γ is the space of all local geodesics (ie maps which are locally isometric) in $Y = X/\Gamma$ which is, in fact, the natural definition for GY.

Since the action of Γ on GX is properly discontinuous, each $\Gamma-$orbit is a closed subset of GX (cf [16, Th. 5.3.4]). Using this, the distance function

$$d_{GY} : GY \times GY \to \mathbb{R}$$

defined by the formula

$$d_{GY}\left(\Gamma g, \Gamma f\right) := \inf \left\{ d\left(x, y\right) \mid x \in \Gamma g, y \in \Gamma f \right\}$$

becomes a metric on GY (cf [16, Th. 6.5.1]). The topology induced by the metric on GY coincides with the quotient topology on GY (see [16, Th. 6.5.2]). Moreover, it can be shown easily that the compact open topology on GY coincides with the quotient topology. Define the geodesic flow on GY by the map

$$\mathbb{R} \times GY \to GY : (t, g) \to t \cdot g$$

where $t\cdot g = p\big(t\cdot\overline{g}\big)$ and $\overline{g}$ is any lift of g in GX. It is easy to check that this definition does not depend on the choice of the lift $\overline{g}$.

Since ∂X is compact and $\partial^2 X$ is open subset of $\partial X \times \partial X$, $\partial^2 X$ is separable. Moreover, GX being, by (4), homeomorphic to $\partial^2 X \times \mathbb{R}$, is also a separable metric space. Thus, its continuous image GY is separable, hence, the metric space

$$GY \textit{ is } 2^{nd} \textit{ countable} \tag{10}$$

1.3 A counter-example

We conclude this section by an example in which all assumptions, except the boundary connectivity condition, of theorem 2 above hold and the geodesic flow is not topologically mixing. Moreover, neither property (1), which is necessary for the proof of mixing presented in this paper, nor property (2), which implies (1), holds.

Example 5 *Let Y be a plane graph homeomorphic to the circle S^1, $X \approx \mathbb{R}$ its universal cover and $\Gamma \approx \mathbb{Z}$ the infinite cyclic group acting on X so that $Y = X/\Gamma$. Then the geodesic flow on GY is not topologically mixing.*

Proof Observe first that X is a 0-hyperbolic space, in fact it is the simplest tree, and its boundary is not connected. All other assumptions of theorem 2 are clearly satisfied. Moreover, a sphere of any radius is disconnected (ie property 2 does not hold) and by the remark following lemma 9 below, X

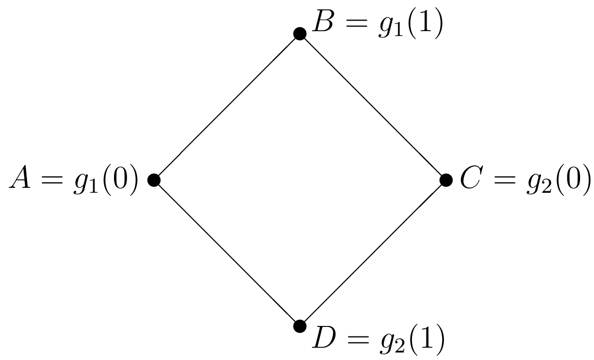

Figure 1:

does not satisfy property (1). We may assume that Y is a plane graph consisting of four vertices denoted A, B, C, D and four edges AB, BC, CD, DA all with length 1 (see figure 1). Let $g_1 : \mathbb{R} \to Y$ be the closed geodesic satisfying $g_1(0) = A$ and $g_1(1) = B$. Similarly, let $g_2 : \mathbb{R} \to Y$ be the closed geodesic with $g_2(0) = C$ and $g_2(1) = D$. Observe that g_1, g_2 are well defined by the above requirements and both have period $\omega = 4$. Consider neighborhoods, in the compact open topology, $\mathcal{O}_1$ and $\mathcal{O}_2$ of g_1 and g_2 respectively, determined by the compact set $[-1/4, 1/4]$ and the number $1/8$ ie

$$f \in \mathcal{O}_1 \Longrightarrow d\big(f(t), g_1(t)\big) < 1/8 \ for\ all\ t \in [-1/8, 1/8]$$

We proceed to show that there exists a sequence $\{t_n\} \subset \mathbb{R}$ converging to $+\infty$ such that $t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ (cf definition 1 above). Let $\{t_n\}$ be the sequence $t_n = 4n + 1$, $n \in \mathbb{N}$.

It is easy to see that if $h \in \mathcal{O}_2$ there exists a number $T \in [-1/8, 1/8]$ such that

$$h(T + t) = g_2(t), \ \forall\ t \in [-1/4, 1/4] \tag{11}$$

Similarly, if $h \in t_n \cdot \mathcal{O}_1$ then there exists a number $S \in [-1/8, 1/8]$ such that

$$h(S + t) = t_n \cdot g_1(t), \ \forall\ t \in [-1/4, 1/4]$$

Observe that $t_n \cdot \mathcal{O}_1 = t_m \cdot \mathcal{O}_1$ for all m, n hence, S does not depend on n. As $t_n = 4n + 1$ and g_1 is periodic with period 4, the latter equation becomes

$$h(S+t) = g_1(1+t), \ \forall \ t \in [-1/4, 1/4] \tag{12}$$

If $h \in t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2$, combining equations (11) and (12) above we obtain that

$$\begin{aligned} 1 &= d(B,C) = d\big(g_1(1+0), g_2(0)\big) = d\big(h(T), h(S)\big) \\ &\leq d\big(h(T), h(0)\big) + d\big(h(0), h(S)\big) = T + S \leq 1/4 \end{aligned}$$

This contradiction shows that $t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ for all $n \in \mathbb{N}$ which completes the proof that the geodesic flow on the circle is not topologically mixing. ▮

Remark In a similar fashion, it can be shown that all statements mentioned for the example above hold for Y being any graph, $\Gamma = \pi_1(Y)$ and X its universal cover.

2 Busemann functions and strong stable sets

In this section we will study Busemann functions (for more details see [2, p.27], [14, Sec.2]) which will be used to define strong stable sets in GX and GY. Throughout this section X will denote a hyperbolic space (i.e. proper geodesic metric space) which, in addition, satisfies conditions (U) and (C). Γ will denote, as before, a non-elementary group of isometries of X. We will consider the product $X \times X \times X$ equipped with the product metric again denoted by d as well as with the product metric induced by the metric d_μ mentioned in proposition 3 above.

Define a function $\alpha : X \times X \times X \to \mathbb{R}$ by letting

$$\alpha(\xi, x, x') := d(x', \xi) - d(x, \xi) \tag{13}$$

for $(\xi, x, x') \in X \times X \times X$. The following calculation shows that α is Lipschitz : if $(\xi, x, x'), (\xi_1, x_1, x_1') \in X \times X \times X$ are arbitrary, we have

$$\begin{aligned}
|\alpha(\xi, x, x') - \alpha(\xi_1, x_1, x_1')| &\leq |d(x_1, \xi_1) - d(x, \xi)| + |d(x', \xi) - d(x_1', \xi_1)| \\
&\leq d(\xi_1, \xi) + d(x_1, x) + d(x_1', x') + d(\xi_1, \xi) \\
&= d\big((\xi, x, x'), (\xi_1, x_1, x_1')\big) + d(\xi_1, \xi) \\
&\leq 2\, d\big((\xi, x, x'), (\xi_1, x_1, x_1')\big)
\end{aligned}$$

In particular, α is uniformly continuous. By proposition 3(a), α is uniformly continuous on the space $X \times X \times X$ equipped with the product metric induced by the metric d_μ. Hence, there exists a unique extension of α to a continuous function defined on the space $(\partial_\mu X \cup X) \times X \times X$. By proposition 3(c) such an extension is continuous on $(\partial X \cup X) \times X \times X$ equipped with its original topology. Thus, we obtain a continuous function

$$(\partial X \cup X) \times X \times X \to \mathbb{R}$$

denoted again by α, called the *generalized Busemann function* which satisfies equation (13) when restricted to $X \times X \times X$.

This function, in fact, generalizes the classical Busemann function whose definition makes sense in our context. To see this, let $\gamma : [0, +\infty) \to X$ be a geodesic ray; the Busemann function associated to γ is a function b_γ on X defined by

$$b_\gamma(x) = \lim_{t\to\infty} [d(x, \gamma(t)) - t]$$

It is easy to see that for any $x \in X$,

$$\begin{aligned}
\alpha\big(\gamma(+\infty), \gamma(0), x\big) &= \lim_{t\to\infty} \alpha\big(\gamma(t), \gamma(0), x\big) \\
&= \lim_{t\to\infty} [d(x, \gamma(t)) - t] = b_\gamma(x)
\end{aligned}$$

and, therefore, the Busemann function b_γ coincides with $\alpha\big(\gamma(+\infty), \gamma(0), \cdot\big)$, ie the restriction of α on $\{\gamma(+\infty)\} \times \{\gamma(0)\} \times X$. Conversely, for arbitrary $\xi \in \partial X$ and $y \in X$, the restriction $\alpha(\xi, y, \cdot) \equiv \alpha|_{\{\xi\}\times\{y\}\times X}$ is simply the Busemann function $b_{\gamma_{y\xi}}$ associated to the unique $\big($cf condition (U)$\big)$ geodesic ray $\gamma_{y\xi}$ with $\gamma_{y\xi}(0) = y$ and $\gamma_{y\xi}(+\infty) = \xi$.

The generalized Busemann function α is Lipschitz with respect to the second and third variable with Lipschitz constant 1. The latter means, in particular, that any Busemann function is Lipschitz with constant 1. To check the Lipschitz property of α, fix $\xi \in \partial X$ and choose a sequence $\{z_n\} \subset X$ such that $z_n \to \xi$. Then for any fixed $x \in X$,

$$\begin{aligned} |\alpha(\xi, y, x) - \alpha(\xi, y', x)| &= \lim_{n\to\infty} |d(x, z_n) - d(y, z_n) - d(x, z_n) + d(y', z_n)| \\ &= \lim_{n\to\infty} |d(y', z_n) - d(y, z_n)| \leq d(y, y') \end{aligned}$$

and, similarly, α can be shown to be Lipschitz with respect to the third variable.

Definition 6 *We say that a geodesic $h \in GX$ belongs to the stable set $W^s(g)$ of a geodesic g if g, h are asymptotic.*
Two points $x, x' \in X$ are said to be equidistant from a point $\xi \in \partial X$ if $\alpha(\xi, x, x') = 0$.
We say that a geodesic $h \in GX$ belongs to the strong stable set $W^{ss}(g)$ of a geodesic g if $h \in W^s(g)$ and $g(0), h(0)$ are equidistant from $g(\infty) = h(\infty)$.
Similarly, if $h, g \in GY$, we say that $h \in W^{ss}(g)$ $\big($respectively $W^s(g)\big)$ if there exist lifts $\overline{h}, \overline{g} \in GX$ of h, g such that $\overline{h} \in W^{ss}(\overline{g})$ $\big($respectively $W^{ss}(\overline{g})\big)$.

The following proposition is a consequence of condition (C) and of the properties of the α function.

Proposition 7 *Let $f, g \in GX$ with $f \in W^{ss}(g)$. Then*

$$\lim_{t\to\infty} d\big(f(t), g(t)\big) = 0.$$

Proof We first show that if $\alpha\big(\xi, f(0), g(0)\big) = 0$, where $\xi = f(+\infty) = g(+\infty)$, then

$$\alpha\big(\xi, f(T), g(T)\big) = 0 \ for\ all\ T \in \mathbb{R} \tag{14}$$

Fix $T > 0$ (we work similarly for $T < 0$). Choose a sequence $\{t_n\} \subset \mathbb{R}$, $t_n > T$ converging to $+\infty$. Then

$$\begin{aligned}
\alpha\big(f(t_n), f(0), g(0)\big) &= d\big(f(t_n), g(0)\big) - d\big(f(t_n), f(0)\big) \\
&= d\big(f(t_n), g(0)\big) - d\big(f(t_n), f(T)\big) - d\big(f(T), f(0)\big) \\
&= \alpha\big(f(t_n), f(T), g(0)\big) - T
\end{aligned}$$

By taking the limits as $t_n \to \infty$ we have, by continuity of a, that

$$\alpha\big(\xi, f(T), g(0)\big) = T$$

In the same way, we have the calculation

$$\begin{aligned}
\alpha\big(g(t_n), f(T), g(0)\big) &= d\big(g(t_n), g(0)\big) - d\big(g(t_n), f(T)\big) \\
&= d\big(g(t_n), g(T)\big) + d\big(g(T), g(0)\big) - d\big(g(t_n), f(T)\big) \\
&= \alpha\big(f(t_n), f(T), g(T)\big) + T
\end{aligned}$$

which implies that $\alpha\big(\xi, f(T), g(T)\big) = \alpha\big(\xi, f(T), g(0)\big) - T = 0$. This completes the proof of equation (14). A repetition of the argument above asserts that for any $s \in \mathbb{R}$,

$$\alpha\big(\xi, f(T), g(T+s)\big) = s \ for\ all\ T \in \mathbb{R} \tag{15}$$

Let now d be the real number posited by Condition (C) making the limit $\lim_{t\to\infty} d\big(f(t), g(t+d)\big)$ equal to 0. We show that $d = 0$ concluding the proof of the proposition. Assume on the contrary that $d \neq 0$. Let T_0 be large enough so that

$$\left|d\big(f(T_0), g(T_0+d)\big)\right| < |d|/2$$

Choose a sequence $\{t_n\} \subset \mathbb{R}$ converging to $+\infty$ with $t_n > T_0$. Then

$$\begin{aligned}
\left|\alpha\big(f(t_n), f(T_0), g(T_0+d)\big)\right| &= \left|d\big(f(t_n), g(T_0+d)\big) - d\big(f(t_n), f(T_0)\big)\right| \\
&\leq \left|d\big(f(T_0), g(T_0+d)\big)\right| < |d|/2
\end{aligned}$$

and by taking the limit as $t_n \to \infty$ we have, by continuity of a, that $\left|\alpha\big(\xi, f(T_0), g(T_0+d)\big)\right| \leq |d|/2$, a contradiction, by equation (15). ∎

We will need the following three lemmata concerning Busemann functions and strong stable sets.

Lemma 8 *Let β a geodesic of X and $x \in X$ arbitrary. Then*
(a) The function $\alpha\big(\beta(+\infty), x, \cdot\big) : \mathrm{Im}\beta \to \mathbb{R}$ is an isometry.
(b) If γ is any geodesic asymptotic with β, then there exists a unique re-parametrization β' of β such that $\alpha\big(\beta(+\infty), \gamma(0), \beta'(0)\big) = 0$, ie $\gamma \in W^{ss}(\beta')$.
(c) Let γ be a geodesic ray in X such that $\beta(-\infty) = \gamma(+\infty)$. Then,

$$\alpha\big(\gamma(+\infty), \gamma(0), \beta(t)\big) = t + \alpha\big(\gamma(+\infty), \gamma(0), \beta(0)\big)$$

In other words, the Busemann function b_γ associated to γ is linear when restricted to $\mathrm{Im}\beta$.

Proof (a) Fix $t, t' \in \mathbb{R}$. Let $\{x_n\} \subset \mathrm{Im}\beta$ be a sequence converging to $\beta(+\infty)$. It is easily shown that for any $x \in X$ and for all n large enough $\big($namely, $\forall\ n$ for which $x_n > \max\{t, t'\}\big)$

$$\begin{aligned} |\alpha(x_n, x, \beta(t)) - \alpha(x_n, x, \beta(t'))| &= |d(x_n, \beta(t)) - d(x_n, \beta(t'))| \\ &= |\beta(t) - \beta(t')| \end{aligned}$$

Using the continuity of the α function and the fact that $x_n \to \beta(+\infty)$ we obtain that $\alpha\big(\beta(+\infty), x, \cdot\big)$ is an isometry on $\mathrm{Im}\beta$.

Part (b) follows from (a) by choosing $x = \gamma(0)$ and then defining $\beta'(t) = \beta(t+T)$ where T is the unique real number such that $\beta(T)$ is the inverse image of 0 via the isometry $\alpha\big(\beta(+\infty), x, \cdot\big)$, ie $\alpha\big(\beta(+\infty), x, \beta(T)\big) = 0$.

(c) Set $\xi = \beta(-\infty) = \gamma(+\infty)$. Using a sequence $\{x_n\}$ converging to ξ and the continuity of the α function it is easily shown that

$$\alpha\big(\xi, \gamma(0), x\big) - \alpha\big(\xi, \beta(0), x\big) = \alpha\big(\xi, \gamma(0), \beta(0)\big),\ \forall\ x \in X$$

Hence, for arbitrary $t \in \mathbb{R}$ we have

$$\alpha\big(\xi, \gamma(0), \beta(t)\big) = \alpha\big(\xi, \beta(0), \beta(t)\big) + \alpha\big(\xi, \gamma(0), \beta(0)\big),$$

Pick $\{t_n\} \subset \mathbb{R}$, with $t_n \to -\infty$. Then,

$$\begin{aligned} \alpha\big(\xi, \beta(0), \beta(t)\big) &= \lim_{n\to\infty} \alpha\big(\beta(t_n), \beta(0), \beta(t)\big) \\ &= \lim_{n\to\infty} [t + |t_n| - |t_n|] = t \end{aligned}$$

This completes the proof of the lemma. ▮

Lemma 9 *Let X be a geodesically complete hyperbolic space with connected boundary. Then for any $x, x' \in X$ with $x \neq x'$ there exists a point $\xi \in \partial X$ such that $\alpha(\xi, x, x') = 0$.*

Proof If $x = x'$ the result is trivial. If $x \neq x'$, extend the geodesic segment joining x with x' to a geodesic, say, g. We may assume that $x = g(s)$, $x' = g(s')$ for some $s, s' \in \mathbb{R}$ with $s < s'$. The continuous function $\alpha(\cdot, x, x')$ restricted to ∂X attains the negative value $\alpha\big(g(+\infty), x, x'\big) = -d(x, x')$ and positive value $\alpha\big(g(-\infty), x, x'\big) = d(x, x')$. As ∂X is assumed connected, the proof of the lemma follows. ∎

Remark If $X = \mathbb{R}$, then $\partial\mathbb{R}$ consists of two points, say $+$ and $-$. It is clear that for any $x, y \in \mathbb{R}$, with $x < y$ we have $\alpha(+, x, y) = x - y$ and $\alpha(-, x, y) = y - x$. Hence, in the case $X = \mathbb{R}$ (cf example 5) the α function does not satisfy property (1).

Lemma 10 *(a) For any $g \in GY$ and $c \in \mathbb{R}$, $\overline{W^{ss}(c \cdot g)} = c \cdot \left(\overline{W^{ss}(g)}\right)$.*

(b) Let $h_1, g_1 \in GY$ with $h_1 \in W^{ss}(g_1)$ and $\mathcal{O}_1 \subset GY$ an open set containing h_1. Then there exists an open set $\mathcal{A}_1$ containing g_1 such that for any $g \in \mathcal{A}_1$, $W^{ss}(g) \cap \mathcal{O}_1 \neq \emptyset$.

(c) If $h \in \overline{W^{ss}(g)}$ then $\overline{W^{ss}(h)} \subset \overline{W^{ss}(g)}$.

Proof (a) If $h \in \overline{W^{ss}(c \cdot g)}$ there exist a sequence $\{h_n\}_{n \in \mathbb{N}} \subset W^{ss}(c \cdot g)$ with $h_n \to h$. It is clear from the definitions that $(-c) \cdot h_n \to (-c) \cdot h$ and $\{(-c) \cdot h_n\}_{n \in \mathbb{N}} \subset W^{ss}(g)$. This shows that $(-c) \cdot h \in \overline{W^{ss}(g)}$ and, hence, $h = c \cdot \big((-c) \cdot h\big) \in c \cdot \left(\overline{W^{ss}(g)}\right)$. Similarly we show the converse inclusion.

(b) The trivialization $H : GX \to \partial^2 X \times \mathbb{R}$ described in section 1.1 above (see equation (4)) maps a geodesic $f \in GX$ to a triple where the third coordinate is a real number. We will be denoting this real number by s_f, ie

$$H(f) = \big(f(-\infty), f(+\infty), s_f\big) \tag{16}$$

Lift g_1, h_1 to geodesics $\overline{g_1}, \overline{h_1} \in GX$ and consider an open neighborhood $\overline{\mathcal{O}_1}$ of $\overline{h_1}$ of the form

$$\overline{\mathcal{O}_1} := H^{-1}\big(O_1 \times O_1' \times \left(s_{\overline{h}_1} - \varepsilon_1', s_{\overline{h}_1} + \varepsilon_1'\right)\big)$$

(cf equation (4) above), where O_1, O_1' are open neighborhoods of $\overline{h_1}(+\infty), \overline{h_1}(-\infty)$ (respectively) in ∂X with $O_1 \cap O_1' = \emptyset$ and ε_1' positive real, all chosen so that

$$p\left(\overline{\mathcal{O}_1}\right) \subseteq \mathcal{O}_1$$

Claim: We may choose $\varepsilon_1 > 0$ and distinct open neighborhoods $A_1, A_1' \subset \partial X$ of $\overline{g_1}(-\infty), \overline{g_1}(+\infty)$ (respectively) such that the neighborhood

$$\overline{\mathcal{A}_1} := H^{-1}\big(A_1 \times A_1' \times \left(s_{\overline{g}_1} - \varepsilon_1, s_{\overline{g}_1} + \varepsilon_1\right)\big)$$

satisfies the following

$$\forall\, \overline{g} \in \overline{\mathcal{A}_1}\ \exists\, \overline{h} \in \overline{\mathcal{O}_1} \text{ such that } \overline{h} \in W^{ss}(\overline{g})$$

Then, by taking $\mathcal{A}_1 := p\left(\overline{\mathcal{A}_1}\right)$ the proof of the lemma is complete : for, if $g \in \mathcal{A}_1$, there exists $\overline{g} \in \overline{\mathcal{A}_1}$ with $p(\overline{g}) = g$ and, by the claim, there exists $\overline{h} \in \overline{\mathcal{O}_1}$ such that $\overline{h} \in W^{ss}(\overline{g})$. As $p\left(\overline{\mathcal{O}_1}\right) \subseteq \mathcal{O}_1$, the geodesic $h = p\left(\overline{h}\right)$ belongs to $\mathcal{O}_1$ and satisfies $h \in W^{ss}(g)$.

Proof of Claim : Choose closed balls $B\big(\overline{h_1}(0)\big)$, $B\big(\overline{g_1}(0)\big)$ around $\overline{h_1}(0)$, $\overline{g_1}(0)$ respectively, both with radius ε_1'. As X is proper, closed balls are compact sets and so is ∂X. Thus the (continuous) generalized Busemann function α restricted to $\partial X \times B\big(\overline{h_1}(0)\big) \times B\big(\overline{g_1}(0)\big)$ is uniformly continuous. This implies that for the number $\varepsilon_1'/2 > 0$, there exists a compact subset

∂B of ∂X containing $\overline{h_1}(+\infty)$ and a number $\lambda > 0$ such that for all $x, x' \in B\big(\overline{h_1}(0)\big)$ with $d(x, x') < \lambda$ and for all $y, y' \in B\big(\overline{g_1}(0)\big)$ with $d(y, y') < \lambda$ and for all $\xi, \xi' \in \partial B$

$$|\alpha(\xi, x, y) - \alpha(\xi', x', y')| < \varepsilon_1'/2 \tag{17}$$

Fix $\varepsilon_1 < \min\{\lambda/2, \varepsilon_1'/2\}$. We may choose small enough neighborhoods O_2, O_2' containing $\overline{h_1}(-\infty), \overline{h_1}(+\infty)$ respectively, so that if $\overline{h}$ is a geodesic with $\overline{h}(+\infty) \in O_2'$ and $\overline{h}(-\infty) \in O_2$ then a suitable re-parametrization of $\overline{h}$ (called again $\overline{h}$) satisfies

$$d\big(\overline{h}(0), \overline{h_1}(0)\big) < \varepsilon_1$$

We may assume that these neighborhoods O_2, O_2' satisfy the inclusions $O_2 \subset O_1$ and $O_2' \subset O_1' \cap \partial B$. Set

$$\overline{\mathcal{O}_2} = H^{-1}\big(O_2 \times O_2' \times \left(s_{\overline{h}_1} - \varepsilon_1, s_{\overline{h}_1} + \varepsilon_1\right)\big)$$

Then we have

$$\forall\ \overline{h} \in \overline{\mathcal{O}_2} \Longrightarrow d\big(\overline{h}(0), \overline{h_1}(0)\big) < \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1 \tag{18}$$

Moreover, using equation (5) and the fact that $\varepsilon_1 < \varepsilon_1'/2$ we have

$$\forall\ s \in (-\varepsilon_1'/2, \varepsilon_1'/2) \ \ \textit{and} \ \ \forall\ \ \overline{h} \in \overline{\mathcal{O}_2} \Longrightarrow s \cdot \overline{h} \in \overline{\mathcal{O}_1} \tag{19}$$

In a similar fashion, we may choose neighborhoods $A_1 \subset \partial X$ containing $\overline{g_1}(-\infty)$ and $A_1' \subset O_2'$ containing $\overline{g_1}(+\infty) = \overline{h_1}(+\infty)$ such that for the neighborhood

$$\overline{\mathcal{A}_1} = H^{-1}\big(A_1 \times A_1' \times \left(s_{\overline{g}_1} - \varepsilon_1, s_{\overline{g}_1} + \varepsilon_1\right)\big)$$

we have

$$\forall\ \overline{g} \in \overline{\mathcal{A}_1} \Longrightarrow d\big(\overline{g}\,(0)\,,\overline{g_1}\,(0)\big) < 2\varepsilon_1 \tag{20}$$

Let $\overline{g} \in \overline{\mathcal{A}_1}$ be arbitrary. Since $A_1' \subset O_2'$, choose $\overline{h} \in \overline{\mathcal{O}_2}$ with $\overline{h}\,(+\infty) = \overline{g}\,(+\infty)$. Then

$$\begin{array}{ll} d\big(\overline{h}\,(0)\,,\overline{h_1}\,(0)\big) < \lambda & \text{by (18) and the fact that } \varepsilon_1 < \lambda/2 \\ d\big(\overline{g}\,(0)\,,\overline{g_1}\,(0)\big) < \lambda & \text{by (20) and the fact that } \varepsilon_1 < \lambda/2 \\ \overline{h}\,(+\infty)\,,\overline{h_1}\,(+\infty) \in \partial B & \text{by construction} \end{array}$$

The above three equations combined with (17) imply that

$$\left|\alpha\big(\overline{h}\,(+\infty)\,,\overline{h}\,(0)\,,\overline{g}\,(0)\big) - \alpha\big(\overline{h_1}\,(+\infty)\,,\overline{h_1}\,(0)\,,\overline{g_1}\,(0)\big)\right| < \varepsilon_1'/2$$

As $\alpha\big(\overline{h}\,(+\infty)\,,\overline{h_1}\,(0)\,,\overline{g_1}\,(0)\big) = 0$, we have that

$$-\varepsilon_1'/2 < \alpha\big(\overline{h}\,(+\infty)\,,\overline{h}\,(0)\,,\overline{g}\,(0)\big) < \varepsilon_1'/2$$

By lemma 8, there exists a real $s \in (-\varepsilon_1'/2, \varepsilon_1'/2)$ such that

$$\alpha\big(\overline{g}\,(+\infty)\,,s\cdot\overline{h}\,(0)\,,\overline{g}\,(0)\big) = 0.$$

Moreover, by equation (19), $s\cdot\overline{h} \in \overline{\mathcal{O}_1}$. Therefore, $s\cdot\overline{h} \in W^{ss}\,(\overline{g})$ which completes the proof of the claim.

Part (c) follows immediately from part (b). ∎

3 Topological transitivity

The geodesic flow $\mathbb{R} \times GY \to GY$ is said to be topologically transitive if given any open sets $\mathcal{O}$ and $\mathcal{U}$ in GY there exists a sequence $\{t_n\} \subset \mathbb{R}$,

$t_n \to +\infty$ such that $t_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all $n \in \mathbb{N}$. It is apparent that topological mixing implies topological transitivity. However, in the proof of topological mixing in section 4 below we will need a property equivalent to topological transitivity, namely, that $\overline{W^s(f)} = GY$ for any $f \in GY$. In this section we will establish this property without dealing with its equivalence to topological transitivity.

Proposition 11 *Let X be a hyperbolic space satisfying conditions (U) and (C), Γ a non-elementary group of isometries of X and Y the quotient space $Y = X/\Gamma$. Assume that the non-wandering set Ω of the geodesic flow on Y equals GY. Then for any $f \in GY$, $\overline{W^s(f)} = GY$.*

For the proof of the above proposition we will need the following result.

Proposition 12 *Let X, Γ and Y be as above and $\Lambda(\Gamma)$ the limit set of the action of Γ on X. Then, $\Omega = GY$ if and only if $\Lambda(\Gamma) = \partial X$.*

Proof Assume first that $\Omega = GY$ and let $\xi \in \partial X$ arbitrary. We show that $\xi \in \Lambda(\Gamma)$. Choose $\overline{f}$ in GX such that $\overline{f}(+\infty) = \xi$. As $\Omega = GY$, $f := p(\overline{f}) \in \Omega$, hence, there exists sequences $\{f_n\} \subset GY$ and $\{t_n\} \subset \mathbb{R}$ such that $t_n \to +\infty$, $f_n \to f$ and $t_n \cdot f_n \to f$. Set $t_n \cdot f_n =: g_n$ and let $\overline{f_n}, \overline{g_n}$ be lifts of f_n, g_n respectively, such that $\overline{f_n} \to \overline{f}$ and $\overline{g_n} \to \overline{f}$. For each $n \in \mathbb{N}$,

$$p\big(\overline{f_n}(t_n)\big) = g_n(0) = p\big(\overline{g_n}(0)\big)$$

and, hence, there exists $\phi_n \in \Gamma$ such that $\phi_n\big(\overline{f_n}(t_n)\big) = \overline{g_n}(0)$. Therefore, there exists $\{\phi_n\} \subset \Gamma$ such that $\phi_n\big(\overline{f_n}(t_n)\big) \to \overline{f}(0)$. Thus,

$$d\Big(\phi_n^{-1}\big(\overline{f}(0)\big), \overline{f_n}(t_n)\Big) \to 0.$$

Since $\overline{f_n}(t_n) \to \overline{f}(+\infty)$ it follows that $\phi_n^{-1}\big(\overline{f}(0)\big) \to \overline{f}(+\infty) = \xi$. This shows that $\xi \in \Lambda(\Gamma)$.

Assume now that $\Lambda(\Gamma) = \partial X$ and let $f \in GY$ arbitrary. We show that $f \in \Omega$. Let $\overline{f} \in GX$ be a lift of f. As $\big(\overline{f}(-\infty), \overline{f}(+\infty)\big) \in \Lambda(\Gamma) \times \Lambda(\Gamma)$ it follows by equation (7) that there exists a sequence of hyperbolic isometries $\{\phi_n\} \subset \Gamma$ such that $\phi_n(+\infty) \to \overline{f}(+\infty)$ and $\phi_n(-\infty) \to \overline{f}(-\infty)$. Let $\overline{f_n}$ be a geodesic joining $\phi_n(+\infty)$ with $\phi_n(-\infty)$ and parametrize it such that $\overline{f_n}(0) \to \overline{f}(0)$. Hence $\overline{f_n} \to \overline{f}$. Set $t_n = d\Big((\phi_n)^{-n}\left(\overline{f_n}(0)\right), \overline{f_n}(0)\Big)$ and $f_n = p \circ f_n$. Clearly, $f_n \to f$ and $t_n \to +\infty$. Moreover, since $p \circ \phi_n = p$,

$$t_n \cdot f_n = p\big(\phi_n\left(\overline{f_n}\right)\big) = p\left(\overline{f_n}\right) \to p\left(\overline{f}\right) = f$$

This shows that $f \in \Omega$ as required. ∎

Proof of proposition 11 We first claim that

$$\text{for any } \xi \in \partial X, \ \overline{\Gamma\xi} = \partial X \tag{21}$$

Let Fix_h be the set of points in ∂X fixed by hyperbolic elements of Γ, ie $Fix_h = \big\{\phi(+\infty) \bigm| \phi \in \Gamma, \phi \textit{ hyperbolic}\big\}$. As Fix_h is dense in $\Lambda(\Gamma)$ (see [10, Ch.II §4]) and $\Lambda(\Gamma) = \partial X$ (cf proposition 12) it suffices to show that $Fix_h \subseteq \overline{\Gamma\xi}$. Let $\eta \in Fix_h$ be arbitrary. If $\eta = \xi$ we have nothing to show. If $\eta \neq \xi$ then $\eta = \phi(+\infty)$ for some hyperbolic $\phi \in \Gamma$. By equation (6) it follows that $\phi^n(\xi) \to \eta$, hence, $\eta \in \overline{\Gamma\xi}$. This completes the proof of equation (21).

Let now $f, g \in GY$ be arbitrary. We proceed to find a sequence $f_n \in W^s(f)$ such that $f_n \to g$. Lift f, g to geodesics $\overline{f}, \overline{g}$ in GX and let $s_{\overline{g}}$ be the unique real number so that $\left(\overline{g}(-\infty), \overline{g}(+\infty), s_{\overline{g}}\right) = H(\overline{g})$ (cf equation 16). By

equation (21) there exists a sequence $\{\phi_n\} \subset \Gamma$ such that $\phi_n\big(\overline{f}(+\infty)\big) \rightarrow \overline{g}(+\infty)$. Define a sequence $\{\overline{g_n}\} \subset GX$ where each $\overline{g_n}$ is determined by the following three conditions (cf equation 4):

(i) $\overline{g_n}(+\infty) = \phi_n\big(\overline{f}(+\infty)\big)$

(ii) $\overline{g_n}(-\infty)$ is any sequence : $\overline{g_n}(-\infty) \rightarrow \overline{g}(-\infty)$, and

(iii) The parametrazation is chosen by requiring $s_{\overline{g_n}} = s_{\overline{g}}\ \forall\ n$.

In other words, $\overline{g_n} := H^{-1}\Big(\overline{g}(-\infty), \phi_n\big(\overline{f}(+\infty)\big), s_{\overline{g}}\Big)$. It is apparent that $\overline{g_n} \rightarrow \overline{g}$. Define $\overline{f_n} := \phi_n^{-1}(\overline{g_n})$ and set $f_n = p\left(\overline{f_n}\right)$. As $p\left(\overline{f_n}\right) = p(\overline{g_n})$ and $\overline{g_n} \rightarrow \overline{g}$, it follows that $f_n \rightarrow p(\overline{g}) = g$. Moreover,

$$\overline{f_n}(+\infty) = \phi_n^{-1}\big(\overline{g}(+\infty)\big) = \phi_n^{-1}\Big(\phi_n\big(\overline{f}(+\infty)\big)\Big) = \overline{f}(+\infty)$$

thus, $f_n \in W^s(f)$ as required. ∎

Corollary 13 *Under the assumptions of proposition 11 above, if there exists a geodesic g whose strong stable set $W^{ss}(g)$ satisfies $\overline{W^{ss}(g)} = GY$ then $\overline{W^{ss}(f)} = GY$ for any closed geodesic $f \in GY$.*

Proof Let g be a geodesic in GY satisfying $\overline{W^{ss}(g)} = GY$ and let f be a closed geodesic in GY with period, say, ω. By proposition 11, $\overline{W^s(f)} = GY$ hence, $g \in \overline{W^s(f)}$. This means that there exists a sequence $\{g_n\} \subset W^s(f)$ such that $g_n \rightarrow g$. For each $n \in \mathbb{N}$, consider lifts $\overline{g_n}, \overline{f}$ of g_n, f respectively, satisfying $\overline{g_n} \in W^{ss}\left(\overline{f}\right)$ and use lemma 8(b) to obtain a real number t_n such that $t_n \cdot g_n \in W^{ss}(f)$. Each t_n may be expressed in the following way

$$t_n = k\omega + c_n$$

where $k \in \mathbb{Z}$ and $c_n \in [0, \omega)$. By choosing, if necessary, a subsequence we have that $c_n \rightarrow c$ for some $c \in [0, \omega]$. Then $c_n \cdot g_n \rightarrow c \cdot g$ with $c_n \cdot g_n \in W^{ss}(f)$.

This means that for some $c \in [0, \omega]$, $c \cdot g \in \overline{W^{ss}(f)}$. By lemma 10(a), we have $c \cdot \overline{W^{ss}(g)} = \overline{W^{ss}(c \cdot g)}$ and, by lemma 10(c), $\overline{W^{ss}(c \cdot g)} \subset \overline{W^{ss}(f)}$. Thus, $GY = c \cdot GY = c \cdot \overline{W^{ss}(g)} \subset \overline{W^{ss}(f)}$. ∎

4 Proof of topological mixing

For the proof of theorem 2 we follow closely the idea used by Eberlein in [12] in the proof of topological mixing of the geodesic flow on Riemannian manifolds of non-positive curvature. However, since we deal with a more general class of spaces, the difficulties which arise here are of much different nature. As usual, the letter X will denote a geodesically complete hyperbolic space satisfying conditions (U) and (C) with connected boundary ∂X, Γ a non-elementary group of isometries of X and Y the quotient space $Y = X/\Gamma$. Moreover, GY is the quotient metric space GX/Γ as described in section 1.2 above. We first establish the following

Proposition 14 *Let X, Y and Γ be as in theorem 2 above. If $\Omega = GY$ then there exists a geodesic g whose strong stable set $W^{ss}(g)$ satisfies $\overline{W^{ss}(g)} = GY$.*

Proof Following the idea of the proof in [12, Th.5.2] we first show that

$$\forall\, \mathcal{O}, \mathcal{U} \subseteq GY \text{ open},\ \exists\, g \in \mathcal{O} : W^{ss}(g) \cap \mathcal{U} \neq \emptyset. \tag{22}$$

We will use the letter p to denote both projections $X \to Y$ and $GX \to GY$. Let $\mathcal{O}, \mathcal{U} \subseteq GY$ be arbitrary open sets. Using equation (7) and the fact that $\partial X = \Lambda(\Gamma)$ (which follows from the assumption $\Omega = GY$ and proposition

12 above), we may choose $f \in p^{-1}(\mathcal{O})$ and $h \in p^{-1}(\mathcal{U})$ such that

$$\big(f(+\infty), h(+\infty)\big) \in \{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma \ \ is \ \ hyperbolic\}$$

Using lemma 9, let ξ_n be in ∂X such that $\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$. We claim that

$$\xi_n \to f(+\infty) \ \ as \ n \to \infty. \tag{23}$$

To see this assume, on the contrary, that $\{\xi_n\}$ (or, a subsequence of it) converges to $\xi \in \partial X$ with $\xi \neq f(+\infty)$. Let β be a geodesic in X such that $\beta(+\infty) = f(+\infty)$ and $\beta(-\infty) = \xi$. Similarly, let $\beta' \in GX$ such that $\beta'(+\infty) = f(+\infty)$ and $\beta'(-\infty) = h(+\infty)$. Since ϕ translates β', i.e. $\phi\big(\mathrm{Im}(\beta')\big) = \mathrm{Im}(\beta')$, it follows that for any $n \in \mathbb{N}$

$$\mathrm{dist}\big(\phi^n(h(0)), \mathrm{Im}\beta'\big) \leq \mathrm{dist}\big(h(0), \mathrm{Im}\beta'\big)$$

As β and β' are asymptotic, a similar statement holds true for β by using condition (C), namely,

$$\exists\, M \in \mathbb{R} : \mathrm{dist}\big(\phi^n(h(0)), \mathrm{Im}\beta\big) \leq M, \ \ for \ all \ n \in \mathbb{N}$$

For each $n \in \mathbb{N}$, let t_n be a real number realizing the distance in the left hand side of the above equation. By equation (6), is clear that $t_n \to +\infty$. As the generalized Busemann function α is Lipschitz (with Lipschitz constant 1) with respect to the third variable, it follows that for all $n \in \mathbb{N}$,

$$\Big|\alpha\big(\xi, f(0), \beta(t_n)\big) - \alpha\big(\xi, f(0), \phi^n(h(0))\big)\Big| \leq M \tag{24}$$

ξ_n is chosen so that $\phi^n(h(0))$ and $f(0)$ are equidistant from ξ_n, thus,

$$\lim_{n\to\infty}\alpha\big(\xi, f(0), \phi^n(h(0))\big) = \lim_{n\to\infty}\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$$

The latter combined with equation (24) implies that

$$\left|\lim_{n\to\infty}\alpha\big(\xi, f(0), \beta(t_n)\big)\right| \leq M$$

This is impossible by lemma 8(c) and the fact that $t_n \to +\infty$. Thus equation (23) is proved. We next show that

$$\phi^{-n}(\xi_n) \to h(+\infty) \ \ as\ n \to \infty. \tag{25}$$

Assume, on the contrary, that $\phi^{-n}(\xi_n)$ (or, a subsequence of it) converges to $\zeta \in \partial X$ with $\zeta \neq h(+\infty)$. We choose a geodesic β with $\beta(+\infty) = \zeta$ and $\beta(-\infty) = h(+\infty)$ and proceed with the proof exactly as in the previous argument by using the facts that

$$\lim_{n\to\infty}\phi^{-n}(f(0)) = h(+\infty).$$

and

$$\alpha\big(\phi^{-n}(\xi_n), \phi^{-n}(f(0)), h(0)\big) = 0$$

Choose now geodesics $f_n \in GX$, $n \in \mathbb{N}$ such that $f_n(+\infty) = \xi_n$ and $f_n(-\infty) = f(-\infty)$. We may parametrize f_n so that $f_n(0) \to f(0)$. This can be done by requiring $s_{\overline{f_n}} = s_{\overline{f}}$ for all $n \in \mathbb{N}$. $\big($cf equation (16)$\big)$. Similarly, choose $h_n \in GX$ such that $h_n(+\infty) = \xi_n$ and $h_n(-\infty) = \phi^n\big(h(-\infty)\big)$ and parametrize them so that

$$\alpha\big(\xi_n, f_n(0), h_n(0)\big) = 0. \tag{26}$$

It is apparent that for n large enough, $f_n \in p^{-1}(\mathcal{O})$ and $h_n \in W^{ss}(f_n)$. If we show that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ for n large enough, then we would have

$$\begin{aligned}
&p(f_n) \in \mathcal{O} \\
&p(h_n) = p\big(\phi^{-n}(h_n)\big) \in \mathcal{U}, \ \textit{and} \\
&p(h_n) \in W^{ss}\big(p(f_n)\big)
\end{aligned}$$

The above three properties imply that for n large enough, $W^{ss}\big(p(f_n)\big) \cap \mathcal{U} \neq \emptyset$, as required in equation (22). We conclude the proof of the proposition by showing that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$. Using equation (25) above, it is clear that

$$\begin{aligned}
\big(\phi^{-n}(h_n)\big)(+\infty) &= \phi^{-n}\big(h_n(\infty)\big) \\
&= \phi^{-n}(\xi_n) \to h(+\infty) \ \textit{as} \ n \to \infty
\end{aligned} \tag{27}$$

Similarly

$$\begin{aligned}
\big(\phi^{-n}(h_n)\big)(-\infty) &= \phi^{-n}\big(h_n(-\infty)\big) \\
&= \phi^{-n}\big(\phi^n\big(h(-\infty)\big)\big) = h(-\infty) \ \textit{as} \ n \to \infty
\end{aligned} \tag{28}$$

Using equations (27) and (28) we have

$$d\Big(h(0), \operatorname{Im}\phi^{-n}(h_n)\Big) \to 0$$

as $n \to +\infty$, and, therefore,

$$d\Big(\phi^n\big(h(0)\big), \operatorname{Im} h_n\Big) \to 0 \ \textit{as} \ n \to \infty \tag{29}$$

Let $h_n(t_n), t_n \in \mathbb{R}$ be the point on $\operatorname{Im} h_n$ closest to $h_n(0)$ which realizes the distance in equation (29) above. As the function α is Lipschitz with respect to the third variable (with Lipschitz constant 1) we have

$$\left|\alpha\big(\xi_n, f(0), \phi^n\big(h(0)\big)\big) - \alpha\big(\xi_n, f(0), h_n(t_n)\big)\right| \leq d\Big(\phi^n\big(h(0)\big), h_n(t_n)\Big)$$

Using the defining property of ξ_n, i.e. $\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$, it follows that

$$\alpha\big(\xi_n, f(0), h_n(t_n)\big) \to 0 \ as\ n \to \infty$$

Similarly, using the fact that $f_n(0) \to f(0)$ as $n \to \infty$ and the Lipschitz property of α with respect to the second variable we have

$$\alpha\big(\xi_n, f_n(0), h_n(t_n)\big) \to 0 \ as\ n \to \infty$$

Since, by lemma 8(b), there is a unique point on each $\operatorname{Im} h_n$ which is equidistant from $f_n(0)$ with respect to ξ_n, namely, $h_n(0)$ $\big($cf equation (26)$\big)$, it follows that $t_n \to 0$ which, combined with equation (29) implies that

$$d\Big(\phi^n\big(h(0)\big), h_n(0)\Big) \to 0 \ as\ n \to \infty.$$

Therefore, $\phi^{-n}\big(h_n(0)\big) \to h(0)$ as $n \to \infty$. The latter combined with the facts

$$\big(\phi^{-n}(h_n)\big)(+\infty) \to h(+\infty) \ \ as\ n \to \infty$$
$$\big(\phi^{-n}(h_n)\big)(-\infty) \to h(-\infty) \ \ as\ n \to \infty$$

implies that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ which concludes the proof equation (22).

Using now a countable basis $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ for the topology of GY, which exists by equation (10), the proof is completed by a standard topological argument (cf [12, Th.5.2]) which we include here for the readers convenience: If $\mathcal{O}$ is an arbitrary open set in GY then by equation (22) above, there exists $g_1 \in \mathcal{O}$ such that $W^{ss}(g_1) \cap \mathcal{O}_1 \neq \emptyset$. Let $h_1 \in W^{ss}(g_1) \cap \mathcal{O}_1$. By lemma 10, there exists an open set $\mathcal{A}_1$ containing g_1 satisfying

$$W^{ss}(h) \cap \mathcal{O}_1 \neq \emptyset$$

for every $h \in \mathcal{A}_1$. Moreover, we may assume that the closure $\overline{\mathcal{A}_1}$ of $\mathcal{A}_1$ lies in $\mathcal{O}_1$ and is compact. Inductively, a sequence of open sets $\mathcal{A}_i$ and a sequence of geodesics g_i are constructed such that

$$\begin{aligned} &\overline{\mathcal{A}_i} \subset \mathcal{A}_{i-1} \\ &g_i \in \mathcal{A}_i \\ &for\ every\ h \in \mathcal{A}_i, W^{ss}(h) \cap \mathcal{O}_i \neq \emptyset \end{aligned}$$

By the finite intersection property of the compact sets $\overline{\mathcal{A}_i}$ it follows that there exists a $g \in \cap_{i=1}^{\infty} \overline{\mathcal{A}_i}$. As $W^{ss}(g) \cap \mathcal{O}_i \neq \emptyset$ for all i, $\overline{W^{ss}(g)} = GY$. ▮

We will need a point-wise version of topological mixing and a criterion for such property.

Definition 15 *Let h, f be in GY and $\{s_n\}_{n \in \mathbb{N}}$ a sequence converging to $+\infty$ or $-\infty$. We say that h is s_n-mixing with f (notation, $h \sim_{s_n} f$) if for every neighborhoods $\mathcal{O}$,$\mathcal{U}$ in GY of h, f respectively, $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large.*

If $h \sim_{s_n} f$ for some $h, f \in GY$ then using decreasing sequences of open neighborhoods of h and f it is easily shown that for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GY$ such that $h_n \to h$ and $r_n \cdot h_n \to f$. The proof of the converse statement is elementary, hence, the following criterion for the s_n-mixing of h, f holds.

Criterion 16 *If $h, f \in GY$ then $h \sim_{s_n} f$ if and only if for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GY$ such that $h_n \to h$ and $r_n \cdot h_n \to f$.*

Proof of Theorem 2 We will show the following property

$$\forall\ \mathcal{O},\mathcal{U} \subseteq GY \ \mathit{open}, \exists\ t'_0 > 0 : t{\cdot}\mathcal{O} \cap \mathcal{U} \neq \emptyset\ \forall\ t \geq t'_0 \tag{30}$$

Then applying this property to the open sets $-\mathcal{O}, -\mathcal{U}$ we obtain a number

$$t''_0 > 0 : \big(t{\cdot}(-\mathcal{O})\big) \cap (-\mathcal{U}) \neq \emptyset\ \forall\ t \geq t''_0. \tag{31}$$

Setting $t_0 = \max\left\{t'_0, t''_0\right\}$ we have that

$$-\big((-t){\cdot}\mathcal{O} \cap \mathcal{U}\big) = \big(-\left((-t){\cdot}\mathcal{O}\right)\big) \cap (-\mathcal{U}) = \big(t{\cdot}(-\mathcal{O})\big) \cap (-\mathcal{U})\,.$$

By (31) it follows that $-\big((-t){\cdot}\mathcal{O}\cap\mathcal{U}\big) \neq \emptyset$, hence $t{\cdot}\mathcal{O}\cap\mathcal{U} \neq \emptyset$ for all $t \leq -t_0$. This combined with (30) competes the proof of theorem 2. We proceed now to show equation (30). For this it suffices to show that

$$\begin{aligned}&\forall\, h, f \in GY \text{ and } \forall\ \{t_n\} \text{ with } t_n \to \infty, \exists \text{ sub-}\\ &\text{sequence } \{s_n\} \subset \{t_n\} \text{ such that } h \sim_{s_n} f.\end{aligned} \tag{32}$$

Claim : In order to show equation (32) it suffices to consider $f \in GY$ closed and arbitrary $h \in GY$.

Proof of claim : Fix arbitrary $h, f \in GY$ and a sequence $\{t_n\}$ with $t_n \to \infty$. Convergence, as usual, is meant to be uniform convergence on compact sets. However, as GY is a metric space and its topology coincides with the compact open topology (see discussion following proposition 4) we can interpret convergence in terms of distances using the metric d_{GY}. By proposition 12 and equation (7), the set of closed geodesics in Y is dense in GY. Hence, there exists a sequence $\{f_k\}$ of closed geodesics such that $f_k \to f$ as $k \to \infty$. By choosing, if necessary a subsequence of $\{f_k\}$, we may assume that

(a) $d_{GY}(f_k, f) < \frac{1}{k}$, for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, since (32) is assumed to hold for the geodesics h, f_k, we have that there exists a subsequence $\{s_n^k\}_{n\in\mathbb{N}}$ of $\{t_n\}$ such that $h \sim_{s_n^k} f_k$. By criterion 16, for each $k \in \mathbb{N}$, there exists a subsequence $\{r_n^k\}_{n\in\mathbb{N}}$ of $\{s_n^k\}_{n\in\mathbb{N}}$ and a sequence $\{h_n^k\}_{n\in\mathbb{N}} \subset GY$ such that $h_n^k \to h$ and $r_n^k \cdot h_n^k \to f_k$, for $n \to \infty$. For each $k \in \mathbb{N}$, we may choose an integer $n(k)$ (in other words, choose a term $r_{n(k)}^k$ in the sequence $\{r_n^k\}_{n\in\mathbb{N}}$) so that the following hold :

(b) $n(k) \to +\infty$ as $k \to \infty$.

(c) $d_{GY}\left(r_n^k \cdot h_n^k, f_k\right) < \frac{1}{k}$, $\forall\ n \geq n(k)$

(d) $d_{GY}\left(h_n^k, h\right) < \frac{1}{k}$, $\forall\ n \geq n(k)$

Set $q_k = r_{n(k)}^k$, $k \in \mathbb{N}$. By (b) above, $\{q_k\}_{k\in\mathbb{N}}$ tends to $+\infty$ as $k \to \infty$ and is, in fact, a subsequence of $\{t_n\}_{k\in\mathbb{N}}$. We will use again criterion 16 above in order to show that $h \sim_{q_k} f$. For, if $\{q_{k_m}\}$ is a subsequence of $\{q_k\}$ then, by (d), the sequence $\left\{h_{n(k_m)}^{k_m}\right\}_{m=1}^{\infty}$ converges to h as $m \to \infty$. Moreover, by (a) and (c),

$$\begin{aligned} d_{GY}\left(q_{k_m} \cdot h_{n(k_m)}^{k_m}, f\right) &\leq d_{GY}\left(q_{k_m} \cdot h_{n(k_m)}^{k_m}, f_{k_m}\right) + d_{GY}\left(f_{k_m}, f\right) \\ &= d_{GY}\left(r_{n(k_m)}^{k_m} \cdot h_{n(k_m)}^{k_m}, f_{k_m}\right) + d_{GY}\left(f_{k_m}, f\right) \\ &< \frac{1}{k_m} + \frac{1}{k_m} \to 0 \text{ as } m \to \infty. \end{aligned}$$

Thus, $q_{k_m} \cdot h_{n(k_m)}^{k_m} \to f$. We have shown that the subsequence $\{q_k\}_{k\in\mathbb{N}}$ of $\{t_n\}$ fulfills the requirement of criterion 16 above. Hence, $h \sim_{q_k} f$. This completes the proof of the claim.

We now proceed to show equation (32) for f closed. Let f, h and $\{t_n\}$ be given with f closed and $h \in GY$. Let $\omega = \mathit{period}(f)$. Choose a subsequence

$\{s_n\}$ of $\{t_n\}$ of the form $s_n = \omega k_n + c_n$ where $\{k_n\}$ is a sequence of natural numbers converging to $+\infty$ and $\{c_n\}$ is a sequence in $[0, \omega)$ converging to a point $c \in [0, \omega)$. Since $(-c) \cdot f$ is a closed geodesic, by corollary 13 and proposition 14 we have that $\overline{W^{ss}((-c) \cdot f)} = GY$. Thus $h \in \overline{W^{ss}((-c) \cdot f)}$.

Step 1 We first treat the case $h \in W^{ss}((-c) \cdot f)$. Choose lifts $\widetilde{f}, \widetilde{h} \in GX$ of f, g respectively, so that $\widetilde{h} \in W^{ss}\left((-c) \cdot \widetilde{f}\right)$. We show that $s_n \cdot h \to f$ as $n \to \infty$ which clearly implies that $h \sim_{s_n} f$. Let $\varepsilon > 0$ and a compact set $K \subset \mathbb{R}$ be given. As $\widetilde{h} \in W^{ss}\left((-c) \cdot \widetilde{f}\right)$, by proposition 7,

$$\lim_{t \to \infty} d\big(\widetilde{h}(t), \widetilde{f}(t - c)\big) = 0.$$

Choose T' satisfying $d\big(\widetilde{h}(t), \widetilde{f}(t-c)\big) < \varepsilon/2$, $\forall\ t > T'$ and let $T = T' + \min K$. Hence, for all n which satisfy $s_n > T$ we have

$$d\big(\widetilde{h}(s_n + t), \widetilde{f}(s_n + t - c)\big) < \varepsilon/2, \ \ \forall\ t \in K$$

Since the projection $p : X \to Y$ is distance decreasing, we have that for all n large enough

$$\begin{aligned} d\big(s_n \cdot h(t), f(\omega k_n + c_n + t - c)\big) &= d\big(s_n \cdot h(t), f(t + c_n - c)\big) \\ &< \varepsilon/2 \ \ \forall\ t \in K \end{aligned} \tag{33}$$

As $c_n \to c$, it is clear that for all n large enough

$$d\big(f(t), f(t + c_n - c)\big) < \varepsilon/2 \ \ \forall\ t \in K \tag{34}$$

Thus, combining equations (33) and (34) we have

$$d\big(s_n \cdot h(t), f(t)\big) < \varepsilon \ \ \forall\ t \in K$$

Therefore, $s_n \cdot h \to f$ which completes the proof of step 1.

Step 2 Consider the general case $h \in \overline{W^{ss}(c \cdot f)}$. Then there exists a sequence $\{h_k\}_{k\in\mathbb{N}} \subset W^{ss}(c \cdot f)$ such that $h_k \to h$ as $n \to \infty$. In order to show that $h \sim_{s_n} f$, let $\mathcal{O}$,$\mathcal{U}$ in GY be neighborhoods of h, f respectively. By step 1 each h_k satisfies $h_k \sim_{s_n} f$. Since $h_k \to h$, $\mathcal{O}$ is also a neighborhood of h_{k_0}, for some k_0 large enough. As $h_{k_0} \sim_{s_n} f$ it follows that $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for n sufficiently large which implies that $h \sim_{s_n} f$. ∎

5 Application to ideal polyhedra

In this section we apply theorem 2 in the class of $n-$*dimensional complete ideal polyhedra*. Certain properties of this class of spaces, including transitivity of the geodesic flow, have been studied in [6], [7], [8] and [9]. Important examples of ideal polyhedra have appeared in Thurston's work, see [17], [16, Sec.10.3], where $3-$manifolds, which are complements of links and knots in $\mathbb{S}^3$, are constructed by gluing together finitely many ideal tetrahedra. In consequence, these finite volume $3-$manifolds are equipped by a complete hyperbolic structure. Moreover, the $2-$skeleton of these $3-$manifolds are examples of $2-$dimensional ideal polyhedra.

Definition 17 *An $n-$dimensional ideal polyhedron is a locally finite union of ideal hyperbolic $n-$polytopes glued together isometrically along their $(n-1)-$faces with at least two germs of polytopes along each $(n-1)-$face. Moreover, with the induced metric (explained below), an $n-$dimensional ideal polyhedron is required to be complete and have curvature less or equal to -1.*

We note here that in the case $n = 2$ the curvature condition in the above definition can be proved, hence, is redundant (see [8, Prop.1]).
Recall that a geodesic metric space is said to have curvature less than or equal to χ if each $y \in Y$ has a neighborhood V_y such that every geodesic triangle of perimeter strictly less than $\frac{2\pi}{\sqrt{\chi}}$ (=$+\infty$ when $\chi \leq 0$) contained in V_y satisfies $CAT(\chi)$ inequality (see [15] for definitions and basic properties).

If Y is an ideal polyhedron of dimension n then Y is naturally a proper geodesic metric space with distance function given as follows : a broken geodesic from point x to a point y is a map $f : [a, b] \to Y$ with $f(a) = x$, $f(b) = y$ for which there exists a subdivision $a = t_0 < t_1 < \cdots < t_{k+1} = b$ of $[a, b]$ such that for all $i = 1, 2, \ldots, n$ the restriction $f|_{[t_i, t_{i+1}]}$ is a geodesic whose image lies in a single ideal polytope. The length of a broken geodesic f is defined to be

$$\sum_{i=0}^{k} \ell \left(f|_{[t_i, t_{i+1}]} \right) = \sum_{i=0}^{k} |f(t_i) - f(t_{i+1})|$$

where the length inside an ideal polytope is measured with respect to the hyperbolic metric $| \ |$. The distance $d(x, y)$ from x to y is then defined to be the lower bound of the lengths of broken geodesics from x to y. An ideal polyhedron Y is called *finite* if finitely many polytopes are glued together to obtain Y.

The universal covering $\widetilde{Y}$ of Y is a complete ideal polyhedron of dimension n satisfying $CAT(-1)$ inequality (see [15, Cor. 2.11]). In particular, $\widetilde{Y}$ is a $\delta-$hyperbolic space in the sense of Gromov. It is well know that $\widetilde{Y}$ satisfies condition (U) (see, for example, [8, Prop.2]). Moreover, if Y is a finite polyhedron, the non-wandering set Ω of the geodesic flow on Y is

equal to GY (see Cor. 10 in [7]) and $\pi_1(Y)$ is a non-elementary group of isometries acting properly discontinuously on $\widetilde{Y}$ (see Cor. 12 in [7]). We will show in this section that $\widetilde{Y}$ is geodesically complete and satisfies condition (C). Hence we will obtain the following application of theorem 2.

Corollary 18 *Let Y be an $n-$dimensional finite ideal polyhedron. If the boundary $\partial\widetilde{Y}$ of its universal cover is connected, then the geodesic flow on Y is topologically mixing.*

In the special case $n = 2$ we show $\big($see proposition 21(b)$\big)$ that the boundary $\partial\widetilde{Y}$ is, in fact, connected. This establishes topological mixing for an arbitrary $2-$dimensional finite ideal polyhedron.

Corollary 19 *The geodesic flow on any finite $2-$dimensional ideal polyhedron is topologically mixing.*

The following question, which is interesting in its own right, remains to be examined. It is related to topological mixing since an affirmative answer will imply topological mixing of the geodesic flow on finite $n-$dimensional ideal polyhedra.

Question : Is the boundary of the universal cover of a finite $n-$dimensional ideal polyhedron ($n \geq 3$) connected?

We begin by showing that condition (C) holds for a class of spaces much wider than ideal polyherda. We would like to thank Prof. A. Papadopoulos for helpful discussions concerning the proof of geodesic completeness of $\widetilde{Y}$ and connectedness of $\partial\widetilde{Y}$, (see proposition 21 below).

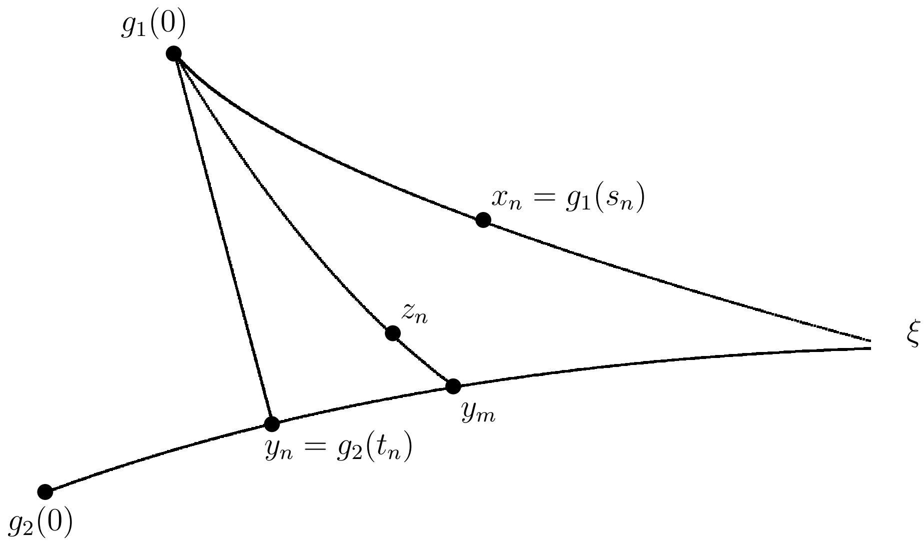

Figure 2:

Proposition 20 *Let Y be a simply connected, locally compact, complete, geodesic metric space Y with curvature less than or equal to χ, $\chi < 0$. Then Y satisfies condition (C).*

Proof Let $g_1, g_2 : [0, \infty) \to Y$ be two asymptotic geodesic rays. Denote by ξ the common boundary point $g_1(+\infty) = g_2(+\infty)$. Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence converging to $+\infty$. For each $n \in \mathbb{N}$, set $y_n = g_2(t_n)$. The sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ given by $s_n = d\big(g_1(0), y_n\big)$ converges to $+\infty$. Denote by x_n the unique point on $\mathrm{Im} g_1$ such that $d\big(g_1(0), x_n\big) = d\big(g_1(0), y_n\big)$, ie $x_n = g_2(s_n)$. For the reader's convenience we have gathered all the above notation in figure 2.

Let $c_n = t_n - s_n$, $n \in \mathbb{N}$. This sequence is increasing and bounded above by $d\big(g_1(0), g_2(0)\big)$. If c is the real number such that $c_n \to c$, we will show that

$$\lim_{t\to\infty} d\big(g_1(t), g_2(t+c)\big) = 0. \qquad (35)$$

We will need the notion of the angle in $CAT-(\chi)$ spaces. We refer the reader to [2, Ch.I Sec.3] for definitions and basic properties. If (x, y, z) is a geodesic triangle in Y, we denote the angle subtended at x by $\measuredangle_x(y, z)$. Recall that if $(\overline{x}, \overline{y}, \overline{z})$ is the comparison triangle in the hyperbolic space $\mathbb{H}^2$ (from now on we assume, without loss of generality, that $\chi = -1$) of the geodesic triangle (x, y, z) then

$$\measuredangle_x(y, z) \leq \measuredangle_{\overline{x}}(\overline{y}, \overline{z})$$

We first show that

$$\measuredangle_{y_n}\big(g_1(0), g_2(0)\big) \to 0 \ \ as \ \ n \to \infty \qquad (36)$$

For each $n \in \mathbb{N}$, let $\left(\overline{g_1(0)}, \overline{g_2(0)}, \overline{y_n}\right)$ be the comparison triangle in the hyperbolic space $\mathbb{H}^2$ of the geodesic triangle $(g_1(0), g_2(0), y_n)$. Since $s_n \to \infty$ and $t_n \to \infty$ as $n \to \infty$, it is not possible to have both angles $\measuredangle_{\overline{g_1(0)}}\big(\overline{g_2(0)}, \overline{y_n}\big)$ and $\measuredangle_{\overline{g_2(0)}}\big(\overline{g_1(0)}, \overline{y_n}\big)$ converging to 0 as $n \to \infty$. Without loss of generality, assume that $\measuredangle_{\overline{g_2(0)}}\big(\overline{g_1(0)}, \overline{y_n}\big)$ is bounded away from zero for all n. Then using the law of cosines

$$\frac{\sinh\, d\big(\overline{g_1(0)}, \overline{g_2(0)}\big)}{\sin\,\left(\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right)\right)} = \frac{\sinh\, s_n}{\sin\,\left(\measuredangle_{\overline{g_2(0)}}\left(\overline{g_1(0)}, \overline{y_n}\right)\right)}$$

it follows that

$$\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to 0 \ or, \ \ \pi \ \ as \ \ n \to \infty$$

If $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to \pi$ then $s_n + t_n \to d\big(\overline{g_1(0)}, \overline{g_2(0)}\big)$ which is impossible (because $\{s_n\}, \{t_n\} \to +\infty$). Thus, $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to 0$. Since $\measuredangle_{y_n}(g_1(0), g_2(0)) \leq \measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right)$ equation (36) is proved.
Our next step is to show that

$$d(x_n, y_n) \to 0 \ \ as \ \ n \to \infty \tag{37}$$

The sequence of geodesic segments $[g_1(0), y_n]$ converges to the geodesic ray g_1 uniformly on compact sets. Thus for each $n \in \mathbb{N}$, we may find $m > n$ such that the neighborhood (in the compact open topology) around g_1 determined by the compact set $[0, s_n]$ and the positive number $1/n$ contains the segment $[g_1(0), y_m]$. In particular, if z_n is the unique point on $[g_1(0), y_m]$ with $d(g_1(0), z_n) = s_n$ we have

$$d(z_n, x_n) < 1/n \tag{38}$$

In order to prove equation (37) above it suffices to show that

$$d(z_n, y_n) \to 0 \ \ as \ \ n \to \infty \tag{39}$$

For each $n \in \mathbb{N}$, let $\left(\overline{g_1(0)}, \overline{y_n}, \overline{y_m}\right)$ be the comparison triangle of the geodesic triangle $(g_1(0), y_n, y_m)$. Let $\overline{z_n}$ be the point corresponding to z_n. Denote by ϕ_n the angles $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{z_n}\right) = \measuredangle_{\overline{z_n}}\left(\overline{g_1(0)}, \overline{y_n}\right)$. Apparently, $\phi_n < \pi/2$ for all n. If $\{\phi_n\}$, or a subsequence, converges to ϕ, for some

$\phi < \pi/2$, then using the facts

$$\measuredangle_{\overline{y_n}} \left(\overline{g_1(0)}, \overline{y_m} \right) \to \pi \ \ \big(by \ \ (36) \big) \ \ and$$
$$\measuredangle_{\overline{z_n}} \left(\overline{g_1(0)}, \overline{y_m} \right) = \pi$$

it follows that

$$\measuredangle_{\overline{y_n}} \left(\overline{z_n}, \overline{y_m} \right) + \measuredangle_{\overline{z_n}} \left(\overline{y_n}, \overline{y_m} \right) > \pi/2 + \pi/2$$

a contradiction. Therefore, $\phi_n \to \pi/2$. Using this and the second law of cosines we obtain that

$$\cosh d \left(\overline{z_n}, \overline{y_m} \right) \to 1 \ \ as \ \ n \to \infty$$

hence, $d\left(\overline{z_n}, \overline{y_n}\right) \to 0$. By comparison, $d\left(z_n, y_m\right) \leq d\left(\overline{z_n}, \overline{y_m}\right)$ which proves equation (39) and, in consequence, proves equation (37).

We proceed now to show equation (35). Since the function

$$t \to d\big(g_1(t), g_2(t+c)\big)$$

is convex with respect to t (see [3, Ch. III]), it suffices to show that for each $\varepsilon > 0$ there exists a positive real number $T = T(\varepsilon)$ such that

$$d\big(g_1(T), g_2(T+c)\big) < \varepsilon$$

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such

$$d\left(x_N, y_N\right) < \varepsilon/2 \ \ and$$
$$\left|c_N - c\right| < \varepsilon/2$$

For the number $T = s_N$ we have

$$\begin{aligned} d\big(g_1(T), g_2(T+c)\big) &\leq d\big(g_1(s_N), g_2(s_N + c_N)\big) + d\big(g_2(s_N + c_N), g_2(s_N + c)\big) \\ &= d\big(g_1(s_N), g_2(t_N)\big) + |c_N - c| \\ &= d(x_N, y_N) + |c_N - c| < \varepsilon \end{aligned}$$

which completes the proof of the proposition. ∎

Proposition 21 *Let Y be an ideal polyhedron of dimension n and $\widetilde{Y}$ its universal cover. Then*
(a) $\widetilde{Y}$ is geodesically complete.
(b) If $n = 2, \partial\widetilde{Y}$ is connected.

Proof (1) Let $g : [0, s) \to \widetilde{Y}$ be a geodesic segment for some $s \in (0, +\infty)$. We may extend such a geodesic from a polytope to an adjacent polytope infinitely many times or, for infinite time inside a single polytope. In the latter case geodesic completeness holds. Hence, we deal with the case in which the image $g\big([0, s)\big)$ either intersects infinitely many ideal polytopes in $\widetilde{Y}$ or it intersects an ideal polytope infinitely many times. It suffices to show that after this extension $s = +\infty$. Assume that $s < \infty$. Then the image $g\big([0, s)\big)$ is contained in a closed ball $B\big(g(0), R\big)$ for some $R > 0$. As $\widetilde{Y}$ is proper, $B\big(g(0), R\big)$ is compact. Therefore, $B\big(g(0), R\big)$ intersects only a finite number of ideal polytopes of $\widetilde{Y}$. Therefore, the image $g\big([0, s)\big)$ must intersect a polytope, say σ, of $\widetilde{Y}$ infinitely many times. Hence, it intersects a face, say $\partial\sigma$, of σ in at least two points, say $g(t_1)$ and $g(t_2)$ with $t_1 < t_2$, such that $\exists\ t \in (t_1, t_2)$, with $g(t) \notin \partial\sigma$. Since $g|_{[t_1,t_2]}$ is a geodesic segment and $g(t_1), g(t_2)$ can be joined by a geodesic lying entirely in $\partial\sigma$, we obtain

two distinct geodesic segments joining $g(t_1), g(t_2)$. This is a contradiction completing the proof of geodesic completeness.

(2) We first show that if Y_0 is an ideal sub-polyhedron of $\widetilde{Y}$ satisfying the property

$$\begin{array}{c}\text{each ideal } 1-\text{simplex of } Y_0 \text{ is adjacent} \\ \text{with exactly two ideal triangles of } Y_0\end{array} \tag{40}$$

then Y_0 is isometric to the hyperbolic 2−space $\mathbb{H}^2$. Since $\widetilde{Y}$ is locally finite it follows that such a subspace Y_0 is a closed subset of $\widetilde{Y}$, hence, Y_0 is a complete space. We next show that Y_0 is simply connected. Assume, on the contrary, that Y_0 is not simply connected. Then there exists a point y_0 and a map $f : [0,1] \to Y_0$ with $f(0) = f(1) = y_0$ so that the closed arc $\gamma = f([0,1])$ is not null homotopic with y_0 fixed. Since Y_0 is locally a $CAT(-1)$ space, there exists a closed arc γ_0 based at y_0 such that γ_0 is locally a geodesic (see [3, Ch.10]). Then γ_0 is also a closed local geodesic arc in $\widetilde{Y}$. As $\widetilde{Y}$ is simply connected, every local geodesic is a geodesic and, hence, γ_0 must be a geodesic which is impossible. It follows that Y_0 is simply connected. It is now clear that Y_0 is isometric to a complete, simply connected manifold of curvature -1 and without boundary. By uniformization theorem, Y_0 is isometric to $\mathbb{H}^2$.

We proceed now to show that the boundary (in the sense of Gromov) $\partial\widetilde{Y}$ of $\widetilde{Y}$ is connected. Let $\xi, \eta \in \partial\widetilde{Y}$ and let γ be the unique (up to parametrization) geodesic with $\gamma(+\infty) = \xi$ and $\gamma(-\infty) = \eta$. Denote by $X^{(1)}$ the 1-skeleton of X. It is clear that γ intersects $X^{(1)}$ transversely. We may assume that $\gamma(0) \notin X^{(1)}$, ie it lies in the interior of an ideal triangle T_0. Let t_1 be the smallest positive real such that $\gamma(t_1) \in X^{(1)}$, and let T_1 be the

unique triangle containing $\gamma(t_1+\varepsilon)$ for ε arbitrarily small. By transversality, $T_0 \neq T_1$. Similarly, a triangle T_2, $T_2 \neq T_1$, is determined by $\mathrm{Im}\gamma$. $\gamma(t_1)$, $\gamma(t_2)$ cannot belong to the same ideal face of T_1, otherwise we would have two distinct geodesic segments joining them. Hence, T_2 is distinct from T_0. In a similar fashion we obtain a sequence $\{T_n\}_{n=-\infty}^{n=+\infty}$ of ideal triangles whose union $Y_0' = \cup_{n=-\infty}^{n=+\infty} T_n$ contains $\mathrm{Im}\gamma$ and so that each triangle T_n in Y_0' has two of its faces identified with a face of T_{n-1} and a face of T_{n+1}. Moreover, these ideal triangles are pair-wise distinct. Starting from the third (free) face of each T_n we may attach infinitely many ideal triangles of $\widetilde{Y}$ in order to obtain an ideal sub-polyhedron Y_0 of $\widetilde{Y}$ containing $\mathrm{Im}\gamma$ and satisfying equation (40). By theorem 2.2 of [11, ch. 3], the isometric embedding $Y_0 \hookrightarrow \widetilde{Y}$ induces a homeomorphism of ∂Y_0 onto its image in $\partial\widetilde{Y}$. Since ∂Y_0 is homeomorphic to S^1, there exists a path in $\partial\widetilde{Y}$ joining ξ with η. ∎

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